

# Stochastic flow for SDEs with jumps and irregular drift term

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**Mathematics Subject Classification (2010):** 60H10, 34F05; 60J75, 35B65.

**Abstract:** We consider non-degenerate SDEs with a  $\beta$ -Hölder continuous and bounded drift term and driven by a Lévy noise  $L$  which is of  $\alpha$ -stable type. If  $\alpha \in [1, 2)$  and  $\beta \in (1 - \frac{\alpha}{2}, 1)$  we show pathwise uniqueness and existence of a stochastic flow. We follow the approach of [Priola, Osaka J. Math. 2012] improving the assumptions on the noise  $L$ . In our previous paper  $L$  was assumed to be non-degenerate,  $\alpha$ -stable and symmetric. Here we can also recover relativistic and truncated stable processes and some classes of tempered stable processes.

## 1 Introduction

We consider the SDE

$$X_t = x + \int_0^t b(X_r) dr + L_t, \quad (1)$$

$x \in \mathbb{R}^d$ ,  $d \geq 1$ ,  $t \geq 0$ , where  $b : \mathbb{R}^d \rightarrow \mathbb{R}^d$  is bounded and Hölder continuous of index  $\beta$ ,  $\beta \in (0, 1)$ , and  $L = (L_t)$  is a non-degenerate  $d$ -dimensional Lévy process of  $\alpha$ -stable type. Our main result gives conditions under which strong uniqueness holds and, moreover, there exists a stochastic flow. The present paper is a continuation of our previous work [21], where (1) has been investigated assuming that  $L$  is non-degenerate,  $\alpha$ -stable and symmetric. Here we can treat more general noises like relativistic and truncated stable processes and some classes of tempered stable processes (see the end of Section 3). We follow the approach in [21] showing that it works in the present more general setting.

There is an increasing interest in pathwise uniqueness for SDEs when  $b$  is singular enough so that the associated deterministic equation (1) with

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$L = 0$  is not well-posed (see [11] and the references therein). An important result in this direction was proved by Veretennikov in [31] (see also [34] for  $d = 1$ ). He was able to prove pathwise (or strong) uniqueness for (1) when  $b : \mathbb{R}^d \rightarrow \mathbb{R}^d$  is only bounded and measurable and  $L$  is a  $d$ -dimensional Wiener process. This theorem has been extended in different directions (see, for instance, [12], [15], [8], [9], [10], [11], [6], [18]).

The situation changes when  $L$  is not a Wiener process but is a symmetric  $\alpha$ -stable process,  $\alpha \in (0, 2)$ . Indeed, when  $d = 1$  and  $\alpha < 1$ , Tanaka, Tsuchiya and Watanabe proved in Theorem 3.2 of [30] that even a bounded and  $\beta$ -Hölder continuous  $b$  is not enough to ensure pathwise uniqueness if  $\alpha + \beta < 1$ . On the other hand, when  $d = 1$  and  $\alpha \geq 1$ , they showed pathwise uniqueness for any bounded and continuous drift term.

Pathwise uniqueness for equation (1) has been proved in [21] for  $d \geq 1$ , when  $L$  is a non-degenerate,  $\alpha$ -stable and symmetric Lévy process (cf. Chapter 3 in [26]), requiring that  $\alpha \in [1, 2)$  and  $b$  is bounded and  $\beta$ -Hölder continuous with  $\beta > 1 - \alpha/2$  (see [33] for an extension of this result when  $b$  belongs to fractional Sobolev spaces and  $\alpha > 1$ ). The approach in [21] differs from the one in [30] and is inspired by [9]. There are two main examples of Lévy processes  $L$  in the class considered in [21]. The first one is the  $\alpha$ -stable process  $L$  having the generator  $\mathcal{L}$  which is the fractional Laplacian  $-(\Delta)^{\alpha/2}$ , i.e.,

$$\mathcal{L}f(x) = \int_{\mathbb{R}^d} (f(x+z) - f(x) - 1_{\{|z| \leq 1\}} z \cdot Df(x)) \frac{\tilde{c}_{\alpha,d}}{|z|^{d+\alpha}} dz, \quad (2)$$

$x \in \mathbb{R}^d$ , where  $f$  is an infinitely differentiable functions with compact support and  $Df(x)$  is the gradient of  $f$  at  $x \in \mathbb{R}^d$ . The second example is  $L = (L_t^1, \dots, L_t^d)$ , where  $(L_t^1), \dots, (L_t^d)$  are independent one-dimensional symmetric stable processes of index  $\alpha$  (see [3] and the references therein). In this case the generator  $\mathcal{L}$  is given by

$$\mathcal{L}f(x) = \sum_{k=1}^d \int_{\mathbb{R}} [f(x + se_k) - f(x) - 1_{\{|s| \leq 1\}} s \partial_{x_k} f(x)] \frac{c_\alpha}{|s|^{1+\alpha}} ds. \quad (3)$$

In the present paper we generalize the class of Lévy processes  $L$  considered in [21] introducing Hypotheses 1 and 2 (see Section 2). The first assumption requires the validity of some gradient estimates for the convolution Markov semigroup  $(R_t)$  associated to  $L$ . This property expresses the fact that  $L$  must be non-degenerate to get the uniqueness result. The second assumption is an integrability condition on the small jumps part of  $L$ . Such assumptions hold not only for the processes  $L$  considered in [21]. Indeed, for instance, the hypotheses are satisfied by truncated stable processes (see [13] and the references therein), by some classes of tempered stable processes (see [24]) and by relativistic stable processes (see, for instance, [5] and [25]).

The following is our main theorem (we recall the definitions of strong solution, pathwise uniqueness and stochastic flow in Definition 5.1).

**Theorem 1.1.** *Let  $L$  be a pure-jump Lévy process satisfying Hypotheses 1 and 2 with some  $\alpha \in [1, 2)$  (see Section 2). Suppose that  $b \in C_b^\beta(\mathbb{R}^d; \mathbb{R}^d)$ , with  $\beta \in (1 - \frac{\alpha}{2}, 1)$ . Then, we have:*

- (i) *Pathwise uniqueness holds for (1), for any  $x \in \mathbb{R}^d$ .*
- (ii) *For any  $x \in \mathbb{R}^d$ , there exists a (unique) strong solution  $(X_t)$  to (1).*
- (iii) *There exists a stochastic flow  $(X_t^x)$  of class  $C^1$ .*

Note that (iii) is stronger than (ii); condition (iii) implies that  $P$ -a.s, for any  $t \geq 0$ , the mapping:  $x \mapsto X_t^x$  is a differentiable homeomorphism from  $\mathbb{R}^d$  onto  $\mathbb{R}^d$  (cf. Section 3 in [16], Section V.7 in [22] and also [23] for the case of log-Lipschitz coefficients). Since  $C_b^\sigma(\mathbb{R}^d, \mathbb{R}^d) \subset C_b^\beta(\mathbb{R}^d, \mathbb{R}^d)$  when  $0 < \beta \leq \sigma$ , our main result holds for any  $\alpha \in [1, 2)$  when  $\beta \in (\frac{1}{2}, 1)$ .

The proof of Theorem 1.1 is given in Section 5 and uses an Itô-Tanaka trick (cf. Section 2 of [9] which considers the case of a Wiener process  $L$ ). Such method requires Schauder estimates for non-local Kolmogorov equations on  $\mathbb{R}^d$  like

$$\lambda v(x) - \mathcal{L}v(x) - b(x) \cdot Dv(x) = g(x), \quad x \in \mathbb{R}^d, \quad (4)$$

with  $\lambda > 0$ . When  $g \in C_b^\beta(\mathbb{R}^d)$ ,  $\alpha \geq 1$  and  $\alpha + \beta > 1$  we obtain a unique solution  $v \in C_b^{\alpha+\beta}(\mathbb{R}^d)$  in Theorem 4.3. By using suitable solutions  $v$  of (4), the Itô-Tanaka trick allows to construct a diffeomorphism  $\psi : \mathbb{R}^d \rightarrow \mathbb{R}^d$ . This mapping  $\psi$  allows to pass from solutions of (1) to solutions of an auxiliary SDE with Lipschitz coefficients (see equation (33)). For such equation the stochastic flow property holds.

The main difficulty of the regularity result for (4) is the case  $\alpha = 1$ . To treat such case we use a localization procedure which is based on Theorem 4.2 where Schauder estimates are proved in the case of  $b(x) = k$ , for any  $x \in \mathbb{R}^d$ , showing that the Schauder constant is independent of  $k$ . Recently, there are many regularity results available for related non-local equations (see, for instance, [3], [2], [4], [19], [7], [28], [19] and the references therein); however our Theorem 4.3 is not covered by such results.

It is an open problem if Theorem 1.1 holds even in the case  $\alpha \in (0, 1)$ . This is mainly due to the difficulty of proving existence of  $C^{\alpha+\beta}$ -solutions to (4) when  $\alpha < 1$  and  $\alpha + \beta > 1$ . However we mention [28] which provides  $C^{\alpha+\beta}$ -regularity results in the case  $0 < \alpha \leq 1$  for  $\mathcal{L} = -(-\Delta)^{\alpha/2}$  using the so-called extension property of the fractional Laplacian (see also Remark 5.5).

The letters  $c$  and  $C$  with subscripts will denote positive constants whose values are unimportant.

## 2 Assumptions and notation

We introduce basic concepts and notation. More details can be found in [1], [26] and [32].

Let  $\langle u, v \rangle$  (or  $u \cdot v$ ) be the euclidean inner product between  $u$  and  $v \in \mathbb{R}^d$ , for any  $d \geq 1$ ; moreover  $|u| = (\langle u, u \rangle)^{1/2}$ . If  $C \subset \mathbb{R}^d$  we denote by  $1_C$  the indicator function of  $C$ . The Borel  $\sigma$ -algebra of  $\mathbb{R}^d$  will be indicated by  $\mathcal{B}(\mathbb{R}^d)$ .

Let us consider a stochastic basis  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$  which satisfies the usual assumptions (see [1, page 72]); the symbol  $E$  will indicate the expectation with respect to  $P$ . Recall that an  $(\mathcal{F}_t)$ -adapted and  $d$ -dimensional stochastic process  $L = (L_t) = (L_t)_{t \geq 0}$ ,  $d \geq 1$ , is a *Lévy process* if it is continuous in probability, it has stationary increments, càdlàg trajectories,  $L_t - L_s$  is independent of  $\mathcal{F}_s$ ,  $0 \leq s \leq t$ , and  $L_0 = 0$ .

One can show (see Chapter 2 in [26]) that there exists a unique triple  $(S, b_0, \nu)$ , where  $S$  is a symmetric non-negative definite  $d \times d$ -matrix,  $b_0 \in \mathbb{R}^d$  and  $\nu$  is a *Lévy measure* (i.e.,  $\nu$  a  $\sigma$ -finite (positive) measure on  $\mathbb{R}^d$  with  $\nu(\{0\}) = 0$  and  $\int_{\mathbb{R}^d} (1 \wedge |y|^2) \nu(dy) < \infty$ ;  $1 \wedge |\cdot| = \min(1, |\cdot|)$ ) such that the characteristic function of  $L_t$  verifies

$$E[e^{i\langle u, L_t \rangle}] = e^{-t\psi(u)} e^{-t\langle Su, u \rangle} e^{it\langle b_0, u \rangle},$$

$$\psi(u) = - \int_{\mathbb{R}^d} (e^{i\langle u, z \rangle} - 1 - i\langle u, z \rangle 1_{\{|z| \leq 1\}}(z)) \nu(dz), \quad u \in \mathbb{R}^d, \quad t \geq 0. \quad (5)$$

The Lévy measure  $\nu$  is also called the intensity jump measure of  $(L_t)$  and (5) is the Lévy-Khintchine formula. The Lévy measure  $\nu$  of a standard  $\alpha$ -stable process  $L$  corresponding to (2) has density  $\frac{c}{|z|^{d+\alpha}}$ ; on the other hand, the Lévy measure  $\nu$  of  $L$  having generator in (3) is concentrated on axes and is singular with respect to the  $d$ -dimensional Lebesgue measure.

In this paper we only deal with a *pure-jump Lévy process  $L$  without drift term*, i.e., we assume that

$$S = 0, \quad b_0 = 0. \quad (6)$$

Note that, possibly changing  $b(x)$  with  $b(x) + b_0$ , in equation (1) we may always assume that  $b_0 = 0$ .

Thanks to (6) we have  $E[e^{i\langle L_t, u \rangle}] = e^{-t\psi(u)}$ ,  $t \geq 0$ ,  $u \in \mathbb{R}^d$ ; the function  $\psi(u)$  is called the *symbol* (or exponent) of the Lévy process  $L$ . Given a symbol  $\psi$  corresponding to a Lévy measure  $\nu$  (see (5)) there exists a unique in law Lévy process  $M = (M_t)$  such that  $E[e^{i\langle M_t, u \rangle}] = e^{-t\psi(u)}$ ,  $t \geq 0$ ,  $u \in \mathbb{R}^d$ .

The convolution Markov semigroup  $(R_t)$  acting on  $C_b(\mathbb{R}^d)$  (the space of all real continuous and bounded functions on  $\mathbb{R}^d$ ) and associated to  $L$  (or to  $\psi$ ) is

$$R_t f(x) = E[f(x + L_t)] = \int_{\mathbb{R}^d} f(x + y) \mu_t(dy), \quad t > 0, \quad f \in C_b(\mathbb{R}^d), \quad (7)$$

$x \in \mathbb{R}^d$ , where  $\mu_t$  is the law of  $L_t$ , and  $R_0 = I$ . Note that the Fourier transform of  $\mu_t$  is  $\hat{\mu}_t(u) = \int_{\mathbb{R}^d} e^{i\langle u, y \rangle} \mu_t(dy) = e^{-t\psi(u)}$ ,  $t \geq 0$ ,  $u \in \mathbb{R}^d$ . The generator  $\mathcal{L}$  of the semigroup  $(R_t)$  is given by

$$\mathcal{L}g(x) = \int_{\mathbb{R}^d} (g(x + y) - g(x) - 1_{\{|y| \leq 1\}} \langle y, Dg(x) \rangle) \nu(dy), \quad (8)$$

$g \in C_c^\infty(\mathbb{R}^d)$ , where  $C_c^\infty(\mathbb{R}^d)$  is the space of all infinitely differentiable functions with compact support (see [1, Section 6.7] and [26, Section 31]);  $Dg(x)$  denotes the gradient of  $g$  at  $x \in \mathbb{R}^d$ .

Before introducing the main assumptions let us recall some function spaces used in the paper. We consider  $C_b(\mathbb{R}^d; \mathbb{R}^k)$ , for integers  $k, d \geq 1$ , as the set of all continuous and bounded functions  $g : \mathbb{R}^d \rightarrow \mathbb{R}^k$ . It is a Banach space endowed with the supremum norm  $\|g\|_0 = \sup_{x \in \mathbb{R}^d} |g(x)|$ ,  $g \in C_b(\mathbb{R}^d; \mathbb{R}^k)$ . Moreover,  $C_b^\beta(\mathbb{R}^d; \mathbb{R}^k)$ ,  $\beta \in (0, 1)$ , is the subspace of all  $\beta$ -Hölder continuous functions  $g$ , i.e.,  $g$  verifies

$$[g]_\beta := \sup_{x \neq x' \in \mathbb{R}^d} \frac{|g(x) - g(x')|}{|x - x'|^\beta} < \infty.$$

$C_b^\beta(\mathbb{R}^d; \mathbb{R}^k)$  is a Banach space with the norm  $\|\cdot\|_\beta = \|\cdot\|_0 + [\cdot]_\beta$ . If  $k = 1$ , we set  $C_b^\beta(\mathbb{R}^d; \mathbb{R}^k) = C_b^\beta(\mathbb{R}^d)$ . Let  $C_b^0(\mathbb{R}^d, \mathbb{R}^k) = C_b(\mathbb{R}^d, \mathbb{R}^k)$  and  $[\cdot]_0 = \|\cdot\|_0$ . For each integer  $n \geq 1$ ,  $\beta \in [0, 1)$ , a function  $g : \mathbb{R}^d \rightarrow \mathbb{R}$  belongs to  $C_b^{n+\beta}(\mathbb{R}^d)$  if  $g \in C^n(\mathbb{R}^d) \cap C_b^\beta(\mathbb{R}^d)$  and, for all  $k \in \{1, \dots, n\}$ , the Fréchet derivatives  $D^k g \in C_b^\beta(\mathbb{R}^d; (\mathbb{R}^d)^{\otimes(k)})$ ;  $C_b^{n+\beta}(\mathbb{R}^d)$  is a Banach space endowed with the norm  $\|g\|_{n+\beta} = \|g\|_0 + \sum_{j=1}^n \|D^j g\|_0 + [D^n g]_\beta$ ,  $g \in C_b^{n+\beta}(\mathbb{R}^d)$ . We also define  $C_b^\infty(\mathbb{R}^d) = \cap_{k \geq 1} C_b^k(\mathbb{R}^d)$ .

*Hypothesis 1.* The Markov semigroup  $(R_t)$  verifies:  $R_t(C_b(\mathbb{R}^d)) \subset C_b^1(\mathbb{R}^d)$ ,  $t > 0$ , and moreover, there exists  $\alpha \in (0, 2)$  and  $c = c_\alpha > 0$  (independent of  $f$  and  $t$ ), such that, for any  $f \in C_b(\mathbb{R}^d)$ ,

$$\sup_{x \in \mathbb{R}^d} |DR_t f(x)| \leq \frac{c}{t^{1/\alpha}} \sup_{x \in \mathbb{R}^d} |f(x)|, \quad t \in (0, 1]. \quad (9)$$

*Hypothesis 2.* For any  $\sigma > \alpha$  ( $\alpha$  is the same as in (9)), it holds

$$\int_{\{|x| \leq 1\}} |x|^\sigma \nu(dx) < \infty. \quad (10)$$

To prove the uniqueness result in [21] it is used that the previous hypotheses hold for non-degenerate symmetric stable processes  $L$ . However, such assumptions are satisfied in more general cases as the next section shows.

*Remark 2.1.* For the sake of completeness, we note that Hypothesis 1 is equivalent to the following two conditions:

- (i) For any  $t > 0$  the measure  $\mu_t$  in (7) has a density  $p_t$  with respect to the Lebesgue measure which belongs to  $C^\infty(\mathbb{R}^d)$ ; moreover  $|Dp_t| \in L^1(\mathbb{R}^d)$ .
- (ii) We have

$$\int_{\mathbb{R}^d} |Dp_t(y)| dy \leq \frac{c}{t^{1/\alpha}}, \quad t \in (0, 1].$$

It is not difficult to check that (i) and (ii) implies Hypothesis 1 (to this purpose one first differentiates the mapping:  $x \mapsto P_t f(x)$  when  $f \in C_c^\infty(\mathbb{R}^d)$ ).

On the other hand, if we have Hypothesis 1 then by the semigroup and the contraction property we deduce that, for any  $t > 0$ ,  $R_t(C_b(\mathbb{R}^d)) \subset C_b^\infty(\mathbb{R}^d)$ . Moreover, (9) implies  $\|D^k R_t f\|_0 \leq \frac{c}{(t \wedge 1)^{1/\alpha}} \|f\|_0$ ,  $t > 0$ ,  $k \geq 1$ ,  $f \in C_b(\mathbb{R}^d)$ . It follows that, for any  $f \in C_c^\infty(\mathbb{R}^d)$ ,  $k \geq 1$ , we have

$|\int_{\mathbb{R}^d} D^k f(y) \mu_t(dy)| \leq c_t \|f\|_0$ , where  $c_t > 0$  is independent of  $f$ . By known properties of the Fourier transform this implies that (i) is satisfied.

To check (ii) we remark that, for  $t \in (0, 1]$ , the estimate

$$|\int_{\mathbb{R}^d} Df(y) p_t(y) dy| = |\int_{\mathbb{R}^d} f(y) Dp_t(y) dy| \leq \frac{c}{t^{1/\alpha}} \|f\|_0,$$

which holds for any  $f \in C_c^\infty(\mathbb{R}^d)$ , implies (ii).

We mention that Theorem 1 of [14] shows that (i) is equivalent to the following Hartman and Wintner condition for the symbol  $\psi$ :

$$\lim_{|y| \rightarrow \infty} \operatorname{Re} \psi(y) \cdot (\log(1 + |y|))^{-1} = \infty.$$

### 3 Classes of Lévy processes satisfying our assumptions

We show examples of Lévy processes which satisfy our hypotheses. In particular, we concentrate on Hypothesis 1.

**Proposition 3.1.** *Suppose that the symbol  $\psi$  of  $L$  (see (5)) can be decomposed as  $\psi(u) = \psi_1(u) + \psi_2(u)$ ,  $u \in \mathbb{R}^d$ , where  $\psi_1$  and  $\psi_2$  are both symbols of Lévy processes and, moreover, the convolution Markov semigroup associated to  $\psi_1$  (see (7)) satisfies gradient estimates (9).*

*Then Hypothesis 1 holds.*

*Proof.* Let  $t \in (0, 1]$ . According to [26, Section 8], there exist unique infinitely divisible Borel probability measures  $\gamma_t^{(1)}$  and  $\gamma_t^{(2)}$  on  $\mathbb{R}^d$  such that the Fourier transform  $\hat{\gamma}_t^{(j)}(z) = e^{-t\psi_j(z)}$ ,  $j = 1, 2$ . Moreover, by [26, Proposition 2.5] we infer that  $\widehat{\gamma_t^{(1)} * \gamma_t^{(2)}} = \hat{\gamma}_t^{(1)} \cdot \hat{\gamma}_t^{(2)} = e^{-t\psi}$ . By the inversion formula we deduce that  $\mu_t = \gamma_t^{(1)} * \gamma_t^{(2)}$  and so (7) can be rewritten as

$$R_t f(x) = \int_{\mathbb{R}^d} \gamma_t^{(1)}(dy) \int_{\mathbb{R}^d} f(x + y + z) \gamma_t^{(2)}(dz),$$

$f \in C_b(\mathbb{R}^d)$ ,  $x \in \mathbb{R}^d$ . Equivalently,  $R_t f(x) = \int_{\mathbb{R}^d} g_t(x + y) \gamma_t^{(1)}(dy)$ , where  $g_t(x) = \int_{\mathbb{R}^d} f(x + z) \gamma_t^{(2)}(dz)$ . By our assumption on  $\psi_1$  it follows that  $R_t f \in C_b^1(\mathbb{R}^d)$  and moreover

$$|DR_t f(x)| \leq \frac{c}{t^{1/\alpha}} \|g_t\|_0 \leq \frac{c}{t^{1/\alpha}} \|f\|_0,$$

where  $c$  is independent of  $t$ ,  $x \in \mathbb{R}^d$  and  $f \in C_b(\mathbb{R}^d)$ . This proves the assertion. ■

The next result follows from Theorem 1.3 in [27].

**Theorem 3.2.** *If for  $\alpha \in (0, 2)$  there exists  $c_1$ ,  $c_2$  and  $M > 0$  such that*

$$c_1 |y|^\alpha \leq \operatorname{Re} \psi(y) \leq c_2 |y|^\alpha, \quad |y| > M, \quad (11)$$

*then Hypothesis 1 holds.*

Condition (11) concerns the “small jump part” of the Lévy process  $L$ . Indeed if  $\psi^{(1)}(u) = -\int_{\{|y|\leq 1\}} (e^{i\langle u, y \rangle} - 1 - i\langle u, y \rangle) \nu(dy)$  and  $\psi^{(2)} = \psi - \psi^{(1)}$ , then  $\psi^{(2)}$  is a bounded function on  $\mathbb{R}^d$  and so (11) holds for  $\psi$  if and only if it holds for  $\psi^{(1)}$ .

Using Theorem 3.2 and Proposition 3.1 (cf. Remark 1.2 of [27]) one can obtain the following generalization of the previous result:

**Proposition 3.3.** *Assume that the Lévy measure  $\nu$  in (5) verifies:  $\nu(A) \geq \nu_1(A)$ ,  $A \in \mathcal{B}(\mathbb{R}^d)$ , where  $\nu_1$  is a Lévy measure on  $\mathbb{R}^d$  such that its corresponding symbol*

$$\psi_1(u) = -\int_{\mathbb{R}^d} (e^{i\langle u, y \rangle} - 1 - i\langle u, y \rangle 1_{\{|y|\leq 1\}}(y)) \nu_1(dy),$$

*verifies (11). Then Hypothesis 1 holds.*

*Proof.* Because the measure  $\nu_2 = \nu - \nu_1$  is still a Lévy measure one can consider its corresponding symbol  $\psi_2 = \psi - \psi_1$ . Applying Proposition 3.1 and Theorem 3.2 we get the assertion. ■

*Examples 3.4.* As in Example 1.5 of [27] let  $\mu$  be a finite non-negative measure on  $\mathcal{B}(\mathbb{R}^d)$  with support on the unit sphere  $S$  and suppose that  $\mu$  is non-degenerate (i.e., its support is not contained in a proper linear subspace of  $\mathbb{R}^d$ ). Let  $r > 0$  and define, for  $A \in \mathcal{B}(\mathbb{R}^d)$ ,

$$\tilde{\nu}(A) = \int_0^r \frac{ds}{s^{1+\alpha}} \int_S 1_A(s\xi) \mu(d\xi). \quad (12)$$

It is not difficult to check that  $\tilde{\nu}$  is a Lévy measure on  $\mathbb{R}^d$ . Moreover  $\tilde{\nu}$  verifies Hypothesis 2. Indeed if  $\sigma > \alpha$  we have  $\int_{\{|x|\leq 1\}} |x|^\sigma \tilde{\nu}(dx) \leq \frac{1}{\sigma-\alpha} \int_S |\xi|^\sigma \mu(d\xi) < \infty$ . In addition the corresponding symbol  $\tilde{\psi}$  verifies the assumptions of Theorem 3.2 and so Hypothesis 1 holds for the convolution Markov semi-group associate to  $\tilde{\psi}$ . Thus Hypotheses 1 and 2 hold in this case.

More generally, applying Proposition 3.3, we obtain that Hypotheses 1 holds if the Lévy measure  $\nu$  of the process  $L$  verifies

$$\nu(A) \geq \tilde{\nu}(A), \quad A \in \mathcal{B}(\mathbb{R}^d).$$

According to the previous discussion *the next examples of Lévy processes verify Hypotheses 1 and 2.*

-  $L$  is a non-degenerate symmetric  $\alpha$ -stable process (see, for instance, [26] and [21]).

In this case  $\nu(A) = \int_0^\infty \frac{ds}{s^{1+\alpha}} \int_S 1_A(s\xi) \mu(d\xi)$ ,  $A \in \mathcal{B}(\mathbb{R}^d)$ ,  $\alpha \in (0, 2)$ , where  $\mu$  is a finite, symmetric measure with the support on the unit sphere  $S$  which is non-degenerate.

-  $L$  is a truncated stable process (see [13] and the references therein).

In this case

$$\nu(A) = c \int_{\{|x|\leq 1\}} \frac{1_A(x)}{|x|^{d+\alpha}} dx, \quad A \in \mathcal{B}(\mathbb{R}^d), \quad \alpha \in (0, 2).$$

Note that this Lévy measure is as  $\tilde{\nu}$  in (12) with  $r = 1$  and  $\mu$  which is the normalized surface measure on  $S$ .

-  $L$  is a tempered stable process of special form (cf. [24]).

We consider

$$\nu(A) = \int_0^\infty \frac{e^{-s} ds}{s^{1+\alpha}} \int_S 1_A(s\xi) \mu(d\xi), \quad A \in \mathcal{B}(\mathbb{R}^d),$$

where  $\mu$  is as in (12),  $\alpha \in (0, 2)$ . Note that  $\nu(A) \geq \tilde{\nu}(A)$ ,  $A \in \mathcal{B}(\mathbb{R}^d)$ , where  $\tilde{\nu}$  is given in (12) with  $r = 1$ .

-  $L$  is a relativistic stable process (cf. [5], [25] and the references therein).

Here  $\psi(u) = (|u|^2 + m^{\frac{2}{\alpha}})^{\frac{\alpha}{2}} - m$ , for some  $m > 0$ ,  $\alpha \in (0, 2)$ ,  $u \in \mathbb{R}^d$ . By Theorem 3.2 it is easy to see that Hypothesis 1 holds. Moreover also Hypothesis 2 is satisfied (see Lemma 2 in [25]).

## 4 Analytic results for the associated Kolmogorov equation

Here we establish existence of regular solutions to equation (4). This will be done through Schauder estimates. Such regular solutions will be used to prove uniqueness for the SDE (1) in Section 5.

*It is important to remark that Hypothesis 2 implies that  $\mathcal{L}g(x)$  in (8) is well defined for any  $g \in C_b^{1+\gamma}(\mathbb{R}^d)$  if  $1 + \gamma > \alpha$ ,  $\gamma \geq 0$ .*

Indeed  $\mathcal{L}g(x)$  can be decomposed into the sum of two integrals, over  $\{|y| > 1\}$  and over  $\{|y| \leq 1\}$  respectively. The first integral is finite since  $g$  is bounded. To treat the second one, we can use the estimate

$$\begin{aligned} & |g(y+x) - g(x) - y \cdot Dg(x)| \\ & \leq \int_0^1 |Dg(x+ry) - Dg(x)| |y| dr \leq [Dg]_\gamma |y|^{1+\gamma}, \quad |y| \leq 1. \end{aligned} \tag{13}$$

In addition  $\mathcal{L}g \in C_b(\mathbb{R}^d)$  if  $g \in C_b^{1+\gamma}(\mathbb{R}^d)$  and  $1 + \gamma > \alpha$ .

We need the following maximum principle (the proof is the same as in Proposition 3.2 of [21]). We have to assume only Hypothesis 2 (see the discussion above).

**Proposition 4.1.** *Assume Hypothesis 2 and consider  $b \in C_b(\mathbb{R}^d, \mathbb{R}^d)$ . If  $v \in C_b^{1+\gamma}(\mathbb{R}^d)$ , with  $1 + \gamma > \alpha$ ,  $\gamma \geq 0$ , is a solution to  $\lambda v - \mathcal{L}v - b \cdot Dv = g$ , with  $\lambda > 0$  and  $g \in C_b(\mathbb{R}^d)$ , then*

$$\|v\|_0 \leq \frac{1}{\lambda} \|g\|_0, \quad \lambda > 0. \tag{14}$$

Next we prove Hölder regularity for (4) when  $b$  is constant, i.e.,  $b(x) = k$ ,  $x \in \mathbb{R}^d$ . The general case of  $b$  Hölder continuous will be treated in Theorem 4.3. We stress that the constant  $c$  in (16) is independent of  $b = k$ .

We impose the natural condition  $\alpha + \beta > 1$  which is needed to get a regular  $C^1$ -solution  $v$ . On the other hand, the assumption  $\alpha + \beta < 2$  is not necessary in the next result. This condition simplifies the proof and it is



not restrictive in the study of uniqueness for (1). Indeed since  $C_b^\sigma(\mathbb{R}^d, \mathbb{R}^d) \subset C_b^\beta(\mathbb{R}^d, \mathbb{R}^d)$  when  $0 < \beta \leq \sigma$ , it is enough to study uniqueness when  $\beta$  satisfies  $\beta + \alpha < 2$ .

**Theorem 4.2.** *Assume Hypotheses 1 and 2. Let  $\beta \in (0, 1)$  with  $\alpha + \beta \in (1, 2)$ . Then, for any  $\lambda > 0$ ,  $k \in \mathbb{R}^d$ ,  $f \in C_b^\beta(\mathbb{R}^d)$ , there exists a unique solution  $v = v_\lambda \in C_b^{\alpha+\beta}(\mathbb{R}^d)$  to the equation*

$$\lambda v - \mathcal{L}v - k \cdot Dv = f \quad (15)$$

on  $\mathbb{R}^d$ . In addition, for any  $\omega > 0$  there exists  $c = c(\omega) > 0$  independent of  $f$ ,  $v$  and  $k$  such that

$$\lambda^{\frac{\alpha+\beta-1}{\alpha}} \|Dv\|_0 + [Dv]_{\alpha+\beta-1} \leq c \|f\|_\beta, \quad \lambda \geq \omega. \quad (16)$$

*Proof.* If  $v \in C_b^{\alpha+\beta}(\mathbb{R}^d)$  with  $\alpha + \beta > 1$  then equation (15) has a meaning thanks to (13). Moreover, uniqueness of solutions is a consequence of Proposition 4.1.

The proof basically follows the one of Theorem 3.3 in [21]. We only indicate some changes. To this purpose note that in Theorem 3.3 of [21] we have (16) for any  $\lambda > 0$  with  $c$  independent of  $\lambda$  since for the Lévy processes considered in [21] gradient estimates (9) hold for any  $t > 0$  (not only for  $t \in (0, 1]$ ).

We first consider the Markov semigroup  $(P_t)$  acting on  $C_b(\mathbb{R}^d)$  and having  $\mathcal{L} + k \cdot D$  as generator, i.e.,

$$P_t g(x) = \int_{\mathbb{R}^d} g(x + y + tk) \mu_t(dy), \quad t > 0, \quad g \in C_b(\mathbb{R}^d),$$

$x \in \mathbb{R}^d$ , where  $\mu_t$  is the law of  $L_t$ , and  $P_0 = I$ . Then we introduce  $v = v_\lambda \in C_b(\mathbb{R}^d)$ ,  $\lambda > 0$ ,

$$v(x) = \int_0^\infty e^{-\lambda t} P_t f(x) dt, \quad x \in \mathbb{R}^d. \quad (17)$$

We will prove that  $v$  belongs to  $C_b^{\alpha+\beta}(\mathbb{R}^d)$  and that (16) holds. Finally we will show that  $v$  solves (15).

*I Part.* Using Hypothesis 1 we prove that  $v \in C_b^{\alpha+\beta}(\mathbb{R}^d)$  and that (16) holds. Note that  $\lambda \|v\|_0 \leq \|f\|_0$ ,  $\lambda > 0$ , since each  $P_t$  is a linear contraction. Then, we remark that

$$P_t g(x) = R_t(g(\cdot + tk))(x), \quad t > 0, \quad g \in C_b(\mathbb{R}^d), \quad x \in \mathbb{R}^d,$$

where  $(R_t)$  is defined in (7). Using Hypothesis 1 we obtain for  $t \in (0, 1]$ ,

$$\sup_{x \in \mathbb{R}^d} |DP_t g(x)| \leq \frac{c \|g(\cdot + kt)\|_0}{t^{1/\alpha}} = \frac{c \|g\|_0}{t^{1/\alpha}}. \quad (18)$$

By using the semigroup and the contraction property of  $(P_t)$  we get easily

$$\sup_{x \in \mathbb{R}^d} |DP_t g(x)| \leq \frac{c \|g\|_0}{(t \wedge 1)^{1/\alpha}}, \quad t > 0, \quad g \in C_b(\mathbb{R}^d). \quad (19)$$

Now interpolation theory ensures that  $(C_b(\mathbb{R}^d), C_b^1(\mathbb{R}^d))_{\beta, \infty} = C_b^\beta(\mathbb{R}^d)$ ,  $\beta \in (0, 1)$ , see, for instance, Chapter 1 in [17]; interpolating estimate (19) with the estimate  $\|DP_t g\|_0 \leq \|Dg\|_0$ ,  $t \geq 0$ ,  $g \in C_b^1(\mathbb{R}^d)$ , we obtain

$$\|DP_t g\|_0 \leq \frac{c_1}{(t \wedge 1)^{(1-\beta)/\alpha}} \|g\|_\beta, \quad t > 0, \quad g \in C_b^\beta(\mathbb{R}^d), \quad (20)$$

with  $c_1 = c_1(c_0, \beta)$ . Similarly, we get

$$\|D^2 P_t g\|_0 \leq \frac{c_2}{(t \wedge 1)^{(2-\beta)/\alpha}} \|g\|_\beta, \quad t > 0, \quad g \in C_b^\beta(\mathbb{R}^d). \quad (21)$$

By (20) (recall that  $\frac{1-\beta}{\alpha} < 1$ ) differentiating under the integral sign in (17) one can easily verify that  $v = v_\lambda$  is differentiable on  $\mathbb{R}^d$ , for  $\lambda > 0$ . Moreover  $Dv$  is bounded on  $\mathbb{R}^d$  and, for any  $\omega > 0$ , there exists  $C_\omega$  such that, for any  $\lambda \geq \omega$  with  $C_\omega > 0$  independent of  $v$ ,  $k$  and  $f$ ,

$$\lambda^{\frac{\alpha+\beta-1}{\alpha}} \|Dv\|_0 \leq C_\omega \|f\|_\beta$$

(we have used that  $\int_0^\infty e^{-\lambda t} (1 \wedge t)^{-\sigma} dt = \frac{c}{\lambda^{1-\sigma}} + \frac{e^{-\lambda}}{\lambda}$ , for  $\sigma < 1$  and  $\lambda > 0$ ).

Finally we have to show that  $Dv \in C_b^\theta(\mathbb{R}^d, \mathbb{R}^d)$ , where  $\theta = \alpha - 1 + \beta \in (0, 1)$ . To this purpose we proceed as in the proof of [2, Proposition 4.2] and [20, Theorem 4.2]. Using (20), we find, for any  $y, y' \in \mathbb{R}^d$ ,  $y \neq y'$ ,  $|y - y'| \leq 1/2$ ,

$$\begin{aligned} |Dv(y) - Dv(y')| &\leq \int_0^{|y-y'|^\alpha} \frac{c\|f\|_\beta}{t^{(1-\beta)/\alpha}} dt + \int_{|y-y'|^\alpha}^1 |DP_t f(y) - DP_t f(y')| dt \\ &\quad + \int_1^\infty e^{-\lambda t} |DP_t f(y) - DP_t f(y')| dt. \end{aligned}$$

Now using (21) we find

$$\begin{aligned} |Dv(y) - Dv(y')| &\leq C\|f\|_\beta \left( |y - y'|^\theta + \int_{|y-y'|^\alpha}^1 \frac{|y - y'|}{t^{(2-\beta)/\alpha}} dt + \frac{e^{-\lambda}}{\lambda} |y - y'| \right) \\ &\leq c_3(\lambda) \|f\|_\beta |y - y'|^\theta, \end{aligned}$$

and so  $[Dv]_{\alpha-1+\beta} \leq c_3(\lambda) \|f\|_\beta$ ,  $\lambda > 0$ , where  $c_3$  is independent of  $f$ ,  $v$ , and  $k$ . Finally to get (16), note that  $c_3(\lambda)$  is decreasing in  $\lambda$ .

*II Part.* We show that  $v$  solves (15), for any  $\lambda > 0$ .

This part can be proved as II Part in the proof of Theorem 3.3 in [21] without changes. The proof is complete.  $\blacksquare$

Next we generalize Theorem 4.2 to the case of  $b \in C_b^\beta(\mathbb{R}^d, \mathbb{R}^d)$ . We can only do this when  $\alpha \geq 1$  (cf. Remark 4.4). To treat the critical case  $\alpha = 1$  we use a localization procedure. This method works since the constant appearing in estimate (16) is independent of  $k \in \mathbb{R}^d$ .

**Theorem 4.3.** *Assume Hypotheses 1 and 2. Let  $\alpha \geq 1$  and  $\beta \in (0, 1)$  be such that  $\alpha + \beta \in (1, 2)$ . Then, for any  $\lambda > 0$ ,  $f \in C_b^\beta(\mathbb{R}^d)$ , there exists a unique solution  $v = v_\lambda \in C_b^{\alpha+\beta}(\mathbb{R}^d)$  to*

$$\lambda v - \mathcal{L}v - b \cdot Dv = f \quad (22)$$

on  $\mathbb{R}^d$ . Moreover, for any  $\omega > 0$ , there exists  $c = c(\omega)$ , independent of  $f$  and  $v$ , such that

$$\lambda \|v\|_0 + [Dv]_{\alpha+\beta-1} \leq c \|f\|_\beta, \quad \lambda \geq \omega. \quad (23)$$

Finally, we have  $\lim_{\lambda \rightarrow \infty} \|Dv_\lambda\|_0 = 0$ .

*Proof.* The proof is the same as the proof of Theorem 3.4 in [21] and uses Theorem 4.2.  $\blacksquare$

*Remark 4.4.* Differently with respect to Theorem 4.2, in Theorem 4.3 we are not able to prove that there exist  $C_b^{\alpha+\beta}$ -solutions to (22) when  $\alpha < 1$ . This problem is clear from the following simple a-priori estimate:  $[Dv]_{\alpha+\beta-1} \leq C \|f\|_\beta + C \|b\|_\beta \|Dv\|_0 + C \|b\|_0 [Dv]_\beta$ . Since  $\alpha < 1$  we only have  $Dv \in C_b^\theta$  with  $\theta = \alpha + \beta - 1 < \beta$  and we cannot continue with the usual analytic methods.

## 5 The main result

We first recall basic facts and notations about Poisson random measures which we will use in the sequel (see also [1], [16], [32]). We will also remind different notions of solutions for (1).

The Poisson random measure  $N$  related to our Lévy process  $L = (L_t)$  (see (1)) is given by

$$N((0, t] \times V) = \sum_{0 < s \leq t} 1_V(\Delta L_s) = \#\{0 < s \leq t : \Delta L_s \in V\},$$

for any Borel set  $V$  in  $\mathbb{R}^d \setminus \{0\}$ , i.e.,  $V \in \mathcal{B}(\mathbb{R}^d \setminus \{0\})$ ,  $t > 0$ . Here  $\Delta L_s = L_s - L_{s-}$  indicates the jump amplitude of  $L$  at  $s > 0$ . The compensated Poisson random measure  $\tilde{N}$  is defined by

$$\tilde{N}((0, t] \times V) = N((0, t] \times V) - t \nu(V)$$

when  $0 \notin \bar{V}$  (by  $\bar{V}$  we denote the closure of  $V$ ); recall that  $\nu$  is given in (5). By our assumption (6) the Lévy-Itô decomposition of the process  $L$  is

$$L_t = \int_0^t \int_{\{|z| \leq 1\}} z \tilde{N}(ds, dz) + \int_0^t \int_{\{|z| > 1\}} z N(ds, dz), \quad t \geq 0. \quad (24)$$

Recall that the stochastic integral  $\int_0^t \int_{\{|z| \leq 1\}} z \tilde{N}(ds, dz)$ ,  $t \geq 0$ , is an  $L^2$ -martingale. The process  $\int_0^t \int_{\{|z| > 1\}} z N(ds, dz) = \sum_{0 < s \leq t, |\Delta L_s| > 1} \Delta L_s$  is a compound Poisson process.

Let  $T > 0$ . The predictable  $\sigma$ -field  $\mathcal{P}$  on  $\Omega \times [0, T]$  is generated by all left-continuous adapted processes (defined on the same stochastic basis on which  $L$  is defined). Let  $V \in \mathcal{B}(\mathbb{R}^d \setminus \{0\})$  and consider a  $\mathcal{P} \times \mathcal{B}(V)$ -measurable mapping  $F : [0, T] \times \Omega \times V \rightarrow \mathbb{R}^d$ .

If  $0 \notin \bar{V}$ , then  $\int_0^T \int_V F(s, x) N(ds, dx) = \sum_{0 < s \leq T} F(s, \Delta L_s) 1_V(\Delta L_s)$  is a random finite sum. If  $E \int_0^T ds \int_V |F(s, x)|^2 \nu(dx) < \infty$ , then one can define the stochastic integral

$$M_t = \int_0^t \int_V F(r, x) \tilde{N}(dr, dx), \quad t \in [0, T]$$

(here we do not need to assume  $0 \notin \bar{V}$ ). The process  $M = (M_t)$  is an  $L^2$ -martingale with a càdlàg modification. By Lemma 2.4 in [16] we have

$$E|M_t|^2 = E \int_0^t dr \int_V |F(r, z)|^2 \nu(dz), \quad t \in [0, T]. \quad (25)$$

*Definition 5.1.* A *weak solution* to (1) is a tuple  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P, L, X)$ , where  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$  is a stochastic basis on which it is defined a pure-jump Lévy process  $L$  (see conditions (5) and (6)) and a càdlàg  $(\mathcal{F}_t)$ -adapted  $\mathbb{R}^d$ -valued process  $X = (X_t) = (X_t)_{t \geq 0}$  which solves (1)  $P$ -a.s..

A weak solution  $X$  which is  $(\bar{\mathcal{F}}_t^L)$ -adapted (here  $(\bar{\mathcal{F}}_t^L)_{t \geq 0}$  denotes the completed natural filtration of  $L$ , i.e., for  $t \geq 0$ ,  $\bar{\mathcal{F}}_t^L$  is the completed  $\sigma$ -algebra generated by  $L_s$ ,  $0 \leq s \leq t$ ) is called *strong solution* (cf. Chapter 3 in [29]).

We say that *pathwise uniqueness* holds for (1) if given two weak solutions  $X$  and  $Y$  (starting at  $x \in \mathbb{R}^d$ ) defined on the same stochastic basis (with respect to the same process  $L$ ) then,  $P$ -a.s.,  $X_t = Y_t$ , for any  $t \geq 0$ .

Given a stochastic basis  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$  on which it is defined a pure-jump Lévy process  $L$ , a *stochastic flow* of class  $C^1$  for (1) is a map  $(t, x, \omega) \mapsto X_t^x(\omega)$ , defined for  $t \geq 0$ ,  $x \in \mathbb{R}^d$ ,  $\omega \in \Omega$  with values in  $\mathbb{R}^d$ , such that

- (a) given  $x \in \mathbb{R}^d$ , the process  $X^x = (X_t^x)_{t \geq 0}$  is a càdlàg  $(\bar{\mathcal{F}}_t^L)$ -adapted solution of (1);
- (b)  $P$ -a.s., for any  $t \geq 0$ , the map  $x \mapsto X_t^x$  is a homeomorphism from  $\mathbb{R}^d$  onto  $\mathbb{R}^d$ ;
- (c)  $P$ -a.s., for any  $t \geq 0$ , the map  $x \mapsto X_t^x$  is a  $C^1$ -function on  $\mathbb{R}^d$ . ■

Starting from a stochastic flow  $(X_t^x)$  one can easily obtain a stochastic flow of Kunita's type  $\xi_{s,t}(x)$ ,  $0 \leq s \leq t$ ,  $x \in \mathbb{R}^d$  (see Section 3.4 in [16] and Remark 4.4 in [21]).

To prove Theorem 1.1 we will consider the following equation on  $\mathbb{R}^d$

$$\lambda u(x) - \mathcal{L}u(x) - Du(x) \cdot b(x) = b(x), \quad x \in \mathbb{R}^d, \quad (26)$$

where  $b : \mathbb{R}^d \rightarrow \mathbb{R}^d$  is given in (1),  $\mathcal{L}$  in (8) and  $\lambda > 0$ ; the equation is intended componentwise, i.e.,  $u : \mathbb{R}^d \rightarrow \mathbb{R}^d$  and

$$\lambda u_j - \mathcal{L}u_j - b \cdot Du_j = b_j, \quad j = 1, \dots, d,$$

with  $u(x) = (u_1(x), \dots, u_d(x))$ ,  $b(x) = (b_1(x), \dots, b_d(x))$ .

The next two results only require that the drift term  $b : \mathbb{R}^d \rightarrow \mathbb{R}^d$  is bounded and continuous. The first lemma provides an Itô-type formula for a solution to (1) (cf. page 15 in [9] where a related result is proved when  $L$  is a Wiener process).

**Lemma 5.2.** *Let  $L$  be a pure-jump Lévy process satisfying Hypothesis 2 for some  $\alpha \in (0, 2)$  and let  $b \in C_b(\mathbb{R}^d, \mathbb{R}^d)$  in (1). Assume that, for some  $\lambda > 0$ , there exists a solution  $u = u_\lambda \in C_b^{1+\gamma}(\mathbb{R}^d, \mathbb{R}^d)$  to (26) with  $\gamma \in [0, 1]$ , and moreover*

$$1 + \gamma > \alpha. \quad (27)$$

*Let  $X = (X_t)$  be a weak solution of (1), such that  $X_0 = x$ ,  $P$ -a.s.. Then,  $P$ -a.s., for any  $t \geq 0$ ,*

$$\begin{aligned} & u(X_t) - u(x) \\ &= x + L_t - X_t + \lambda \int_0^t u(X_r) dr + \int_0^t \int_{\mathbb{R}^d \setminus \{0\}} [u(X_{r-} + x) - u(X_{r-})] \tilde{N}(dr, dx). \end{aligned} \quad (28)$$

*Proof.* To see that the stochastic integral in (28) is well defined we note that

$$\begin{aligned} & E \int_0^t dr \int_{\mathbb{R}^d} |u(X_{r-} + z) - u(X_{r-})|^2 \nu(dz) \\ & \leq 4t \|u\|_0^2 \int_{\{|z|>1\}} \nu(dz) + t \|Du\|_0^2 \int_{\{|z|\leq 1\}} |z|^2 \nu(dz) < \infty. \end{aligned}$$

The assertion is obtained applying Itô's formula to  $u_i(X_t)$ ,  $i = 1, \dots, d$ , as in the proof of Lemma 4.2 in [21] (for more details on Itô's formula see [1, Theorem 4.4.7] and [16, Section 2.3]).

We note that Itô's formula (as it is usually stated) would require that  $u_i \in C^2(\mathbb{R}^d)$ . However in our situation we have

$$\int_{\{|x|\leq 1\}} |x|^{1+\gamma} \nu(dx) < \infty, \quad (29)$$

since  $1 + \gamma > \alpha$  and Hypothesis 2 holds. Using (29), (13) and an approximation argument one proves that, for any  $f \in C_b^{1+\gamma}(\mathbb{R}^d)$ ,  $P$ -a.s.,  $t \geq 0$ ,

$$\begin{aligned} f(X_t) - f(x) &= \int_0^t \int_{\mathbb{R}^d \setminus \{0\}} [f(X_{s-} + z) - f(X_{s-})] \tilde{N}(ds, dz) \\ &+ \int_0^t \mathcal{L}f(X_s) ds + \int_0^t b(X_s) \cdot Df(X_s) ds \end{aligned} \quad (30)$$

(cf. Itô's formula (4.6) in [21]). Thus we can apply Itô's formula to  $u_i(X_t)$ .

Remark that, for any  $i = 1, \dots, d$ , we have  $\mathcal{L}u_i + b \cdot Du_i = \lambda u_i - b_i$  (see (26)). Thus we can substitute in the Itô formula for  $u_i(X_t)$  the term

$$\int_0^t \mathcal{L}u_i(X_r) dr + \int_0^t Du_i(X_r) \cdot b(X_r) dr$$

with  $-\int_0^t b_i(X_r) dr + \lambda \int_0^t u_i(X_r) dr = x_i + (L_t)_i - (X_t)_i + \lambda \int_0^t u_i(X_r) dr$  and obtain the assertion.  $\blacksquare$

Theorem 1.1 will follow from the next result.

**Theorem 5.3.** *Let  $L$  be a pure-jump Lévy process satisfying Hypothesis 2 for some  $\alpha \in (0, 2)$  and let  $b \in C_b(\mathbb{R}^d, \mathbb{R}^d)$  in (1). Suppose that, for some  $\lambda > 0$ , there exists  $u = u_\lambda \in C_b^{1+\gamma}(\mathbb{R}^d, \mathbb{R}^d)$  which solves (26) with  $\gamma \in ]0, 1]$ , and such that  $c_\lambda = \|Du_\lambda\|_0 < 1/3$ . Moreover, suppose that*

$$2\gamma > \alpha. \quad (31)$$

*Then, assertions (i), (ii) and (iii) of Theorem 1.1 hold.*

*Proof.* We stress that  $2\gamma > \alpha$  implies the condition  $1 + \gamma > \alpha$  in Lemma 5.2.

Since  $\|Du\|_0 < 1/3$ , the classical Hadamard theorem (see [22, page 330]) implies that the mapping

$$\psi : \mathbb{R}^d \rightarrow \mathbb{R}^d, \quad \psi(x) = x + u(x), \quad x \in \mathbb{R}^d,$$

is a  $C^1$ -diffeomorphism from  $\mathbb{R}^d$  onto  $\mathbb{R}^d$ . Moreover,  $D\psi^{-1}$  is bounded on  $\mathbb{R}^d$  with  $\|D\psi^{-1}\|_0 \leq \frac{1}{1-c_\lambda} < \frac{3}{2}$  thanks to

$$D\psi^{-1}(z) = [I + Du(\psi^{-1}(z))]^{-1} = \sum_{k \geq 0} (-Du(\psi^{-1}(z)))^k, \quad z \in \mathbb{R}^d,$$

and we have the estimate (see page 444 of [21])

$$|D\psi^{-1}(z) - D\psi^{-1}(z')| \leq c_1 [Du]_\gamma |z - z'|^\gamma, \quad z, z' \in \mathbb{R}^d. \quad (32)$$

Let  $r \in (0, 1]$  to be fixed later (in the proof of (i) and (ii) we may consider  $r = 1$ ) and introduce the following *auxiliary SDE* (cf. (4.11) in [21])

$$Y_t = y + \int_0^t \tilde{b}(Y_s) ds + \int_0^t \int_{\{|z| \leq r\}} g(Y_{s-}, z) \tilde{N}(ds, dz) + \int_0^t \int_{\{|z| > r\}} g(Y_{s-}, z) N(ds, dz), \quad t \geq 0, \quad (33)$$

where

$$\begin{aligned} \tilde{b}(y) &= \lambda u(\psi^{-1}(y)) - \int_{\{|z| > r\}} [u(\psi^{-1}(y) + z) - u(\psi^{-1}(y))] \nu(dz) - \int_{\{r < |z| \leq 1\}} z \nu(dz) \\ &= \lambda u(\psi^{-1}(y)) - \int_{\{|z| > r\}} [g(y, z) - z] \nu(dz) - \int_{\{r < |z| \leq 1\}} z \nu(dz) \end{aligned}$$

and

$$\begin{aligned} g(y, z) &= u(\psi^{-1}(y) + z) + z - u(\psi^{-1}(y)) = \\ &= u(\psi^{-1}(y) + z) + z + \psi^{-1}(y) - \psi^{-1}(y) - u(\psi^{-1}(y)) \\ &= [\psi(\psi^{-1}(y) + z) - y], \quad y \in \mathbb{R}^d, \quad z \in \mathbb{R}^d. \end{aligned}$$

Note that (33) is a SDE which satisfies the *usual Lipschitz conditions* (cf. Section 3.5 of [16] or Section 6.2 in [1]). Indeed  $\tilde{b}$  is a Lipschitz function,  $|g(y, z)| \leq (1 + \|Du\|_0)|z|$ , for each  $y, z \in \mathbb{R}^d$ , and, moreover (see page 442

in [21] and Lemma 4.1 in [21]), for any  $y, y' \in \mathbb{R}^d$  (recall that  $2\gamma > \alpha$  and we are assuming (10)),

$$\begin{aligned} \int_{\{|z| \leq 1\}} |g(y, z) - g(y', z)|^2 \nu(dz) &\leq c_1 \|u\|_{1+\gamma} |y - y'|^2 \int_{\{|z| \leq 1\}} |z|^{2\gamma} \nu(dz) \\ &\leq c_2 |y - y'|^2. \end{aligned}$$

(i) Let  $x \in \mathbb{R}^d$ . To prove pathwise uniqueness for our equation (1) note that if  $(X_t)$  is a weak solution to (1) then using (28) and formula (24) we easily get that  $(\psi(X_t))_{t \geq 0}$  is a (strong) solution to (33) with  $y = \psi(x)$ . Since pathwise uniqueness holds for (33) if we consider two weak solutions  $(X_t)$  and  $(Z_t)$  of (1) (starting at  $x \in \mathbb{R}^d$ ) defined on the same stochastic basis and with respect to the same Lévy process  $L$  then we obtain,  $P$ -a.s.,  $\psi(X_t) = \psi(Z_t)$ ,  $t \geq 0$ , and so,  $P$ -a.s.,  $X_t = Z_t$ ,  $t \geq 0$ .

(ii) Let us fix a stochastic basis on which it is defined a pure-jump Lévy process  $L$  which satisfies our hypotheses.

Let us first consider equation (33) and fix  $r = 1$ . Thanks to the regularity of the coefficients (cf. [16] or [1]), for any  $y \in \mathbb{R}^d$ , on our fixed stochastic basis there exists a unique strong solution  $Y^y = (Y_t^y)$ .

Let  $x \in \mathbb{R}^d$  and set  $Y_t^{\psi(x)} = Y_t$ ,  $t \geq 0$ . If we define  $(X_t) = (\psi^{-1}(Y_t))$  we get that  $(X_t)$  is a *strong solution* to (1) starting at  $x \in \mathbb{R}^d$  by Itô's formula.

Standard Itô's formula (see Theorem 4.4.7 in [1]) says that if  $F \in C^2(\mathbb{R}^d, \mathbb{R}^d)$  then we have,  $P$ -a.s., for  $t \geq 0$ ,

$$\begin{aligned} F(Y_t) &= F(\psi(x)) + \int_0^t DF(Y_s) \tilde{b}(Y_s) ds \\ &+ \int_0^t \int_{\{|z| \leq 1\}} [F(Y_{s-} + g(Y_{s-}, z)) - F(Y_{s-})] \tilde{N}(ds, dz) \\ &+ \int_0^t \int_{\{|z| > 1\}} [F(Y_{s-} + g(Y_{s-}, z)) - F(Y_{s-})] N(ds, dz) \\ &+ \int_0^t ds \int_{\{|z| \leq 1\}} [F(Y_{s-} + g(Y_{s-}, z)) - F(Y_{s-}) - DF(Y_s)g(Y_s, z)] \nu(dz). \end{aligned} \tag{34}$$

Recall that the third line in (34) is a finite random sum. Note also that  $\int_0^t DF(Y_s) \tilde{b}(Y_s) ds$  has components  $\int_0^t DF_i(Y_s) \cdot \tilde{b}(Y_s) ds$ ,  $i = 1, \dots, d$ .

We fix  $t > 0$  and show that the previous formula holds even with  $F = \psi^{-1}$  arguing by approximation (cf. page 438 in [21]). Recall that  $\psi^{-1} \in C^1(\mathbb{R}^d, \mathbb{R}^d)$  and  $D\psi^{-1}$  is bounded on  $\mathbb{R}^d$  and satisfies (32).

By convolution with mollifiers  $(\rho_n)$ , i.e.,  $F_n(x) = \int_{\mathbb{R}^d} \psi^{-1}(y) \rho_n(x+y) dy$ ,  $x \in \mathbb{R}^d$ , and, possibly passing to a subsequence, we find  $(F_n) \subset C^\infty(\mathbb{R}^d, \mathbb{R}^d)$  such that  $F_n \rightarrow \psi^{-1}$  in  $C^{1+\gamma'}(K; \mathbb{R}^d)$ , for any compact set  $K \subset \mathbb{R}^d$  and  $0 < \gamma' < \gamma$ . Moreover, we have the estimate  $\|DF_n\|_0 \leq \|D\psi^{-1}\|_0$ ,  $n \geq 1$ , and, using (32) and (13), also

$$|F_n(y+x) - F_n(y) - DF_n(y)x| \leq [D\psi^{-1}]_\gamma |x|^{1+\gamma}, \quad |x| \leq 1, \quad y \in \mathbb{R}^d, \quad n \geq 1. \tag{35}$$

Let us write Itô' formula (34) with  $F$  replaced by  $F_n$ . We can easily pass to the limit as  $n \rightarrow \infty$  with  $\omega \in \Omega$  fixed in the first, third and fourth line of

(34); for instance, in order to pass to the limit in the fourth line of (34) we use estimate (35) which gives

$$\begin{aligned} & |F_n(Y_{s-} + g(Y_{s-}, z)) - F_n(Y_{s-}) - DF_n(Y_s)g(Y_s, z)| \leq [D\psi^{-1}]_\gamma |g(Y_s, z)|^{1+\gamma} \\ & \leq C|z|^{1+\gamma}, \end{aligned}$$

$|z| \leq 1$ ,  $s \in [0, t]$ ,  $n \geq 1$ , and then we apply the Lebesgue convergence theorem taking into account (29). In the second line of (34) (written with  $F$  replaced by  $F_n$ ) we can pass to the limit in  $L^2(\Omega)$  thanks to the isometry formula (25); indeed we have

$$\begin{aligned} & E \left| \int_0^t \int_{\{|z| \leq 1\}} [\psi^{-1}(Y_{s-} + g(Y_{s-}, z)) - F_n(Y_{s-} + g(Y_{s-}, z)) \right. \\ & \quad \left. - \psi^{-1}(Y_{s-}) + F_n(Y_{s-})] \tilde{N}(ds, dz) \right|^2 \\ & = E \int_0^t ds \int_{\{|z| \leq 1\}} |\psi^{-1}(Y_{s-} + g(Y_{s-}, z)) - F_n(Y_{s-} + g(Y_{s-}, z)) \\ & \quad - \psi^{-1}(Y_{s-}) + F_n(Y_{s-})|^2 \nu(dz) \rightarrow 0, \end{aligned}$$

as  $n \rightarrow \infty$ , thanks to the estimate

$$|\psi^{-1}(Y_{s-} + g(Y_{s-}, z)) - F_n(Y_{s-} + g(Y_{s-}, z)) - \psi^{-1}(Y_{s-}) + F_n(Y_{s-})|^2 \leq C|z|^2,$$

where  $C$  is independent of  $n$ ,  $s$  and  $\omega \in \Omega$  (recall that  $\|DF_n\|_0 \leq \|D\psi^{-1}\|_0$ ,  $n \geq 1$ ). By Itô's formula with  $F = \psi^{-1}$  we get

$$\begin{aligned} & \psi^{-1}(Y_t) = x + \int_0^t D\psi^{-1}(Y_s) \tilde{b}(Y_s) ds \\ & + \int_0^t \int_{\{|z| \leq 1\}} [\psi^{-1}(Y_{s-} + [\psi(\psi^{-1}(Y_{s-}) + z) - Y_{s-}]) - \psi^{-1}(Y_{s-})] \tilde{N}(ds, dz) \\ & \quad + \int_0^t \int_{\{|z| > 1\}} z N(ds, dz) \\ & + \int_0^t ds \int_{\{|z| \leq 1\}} [z - D\psi^{-1}(Y_s)[\psi(\psi^{-1}(Y_{s-}) + z) - Y_{s-}]] \nu(dz). \end{aligned}$$

It follows that

$$\begin{aligned} & \psi^{-1}(Y_t) = x + L_t + \lambda \int_0^t D\psi^{-1}(Y_s) u(\psi^{-1}(Y_s)) ds \\ & - \int_0^t D\psi^{-1}(Y_s) ds \int_{\{|z| > 1\}} ([\psi(\psi^{-1}(Y_{s-}) + z) - Y_{s-}] - z) \nu(dz) \\ & + \int_0^t ds \int_{\{|z| \leq 1\}} [z - D\psi^{-1}(Y_s)[\psi(\psi^{-1}(Y_{s-}) + z) - Y_{s-}]] \nu(dz). \end{aligned}$$



Thus, using that  $\lambda u = \mathcal{L}u + Du b + b$ , we find

$$\begin{aligned} \psi^{-1}(Y_t) &= x + L_t \\ &+ \int_0^t D\psi^{-1}(Y_s) ds \int_{\{|z| \leq 1\}} ([\psi(\psi^{-1}(Y_{s-}) + z) - Y_{s-}] - z - Du(\psi^{-1}(Y_s))z) \nu(dz) \\ &+ \int_0^t D\psi^{-1}(Y_s) Du(\psi^{-1}(Y_s)) b(\psi^{-1}(Y_s)) ds \\ &+ \int_0^t D\psi^{-1}(Y_s) b(\psi^{-1}(Y_s)) ds \\ &+ \int_0^t ds \int_{\{|z| \leq 1\}} [z - D\psi^{-1}(Y_s)[\psi(\psi^{-1}(Y_{s-}) + z) - Y_{s-}]] \nu(dz). \end{aligned}$$

Since  $Du(y) = D\psi(y) - I$  and  $D\psi^{-1}(y)D\psi(\psi^{-1}(y)) = I$ ,  $y \in \mathbb{R}^d$ , we get

$$\psi^{-1}(Y_t) = x + L_t + \int_0^t b(\psi^{-1}(Y_s)) ds.$$

This shows that  $(X_t) = (\psi^{-1}(Y_t))$  is the unique strong solution to (1).

(iii) In order to prove the existence of a stochastic flow for (1) it is enough to establish such property for equation (33) and then use that

$$X_t^x = \psi^{-1}(Y_t^{\psi(x)}). \quad (36)$$

To this purpose one can use Theorems 3.10 and 3.4 in [16]. Checking the validity of their assumptions for equation (33) is quite involved and it is done in the proof of Theorem 4.3 in [21] (one has also to fix  $r$  in (33) small enough; see also page 443 in [21] for the proof of the differentiability property). This completes the proof.  $\blacksquare$

*Remark 5.4.* We point out that the previous proof also provides a formula for the derivative  $DX_t^x$  with respect to  $x$  in terms of  $H_t^y = DY_t^y$  (see (36)). Indeed  $y \mapsto Y_t^y$  is  $C^1$ ,  $P$ -a.s., and Theorem 3.4 in [16] provides the following formula (cf. the formula at the end of page 445 in [21])

$$\begin{aligned} H_t^y &= I + \lambda \int_0^t Du(\psi^{-1}(Y_s^y)) D\psi^{-1}(Y_s^y) H_s^y ds \\ &+ \int_0^t \int_{\mathbb{R}^d \setminus \{0\}} (D_y h(Y_{s-}^y, z) H_{s-}^y) \tilde{N}(ds, dz), \quad t \geq 0, y \in \mathbb{R}^d. \end{aligned}$$

According to [21] the stochastic integral is meaningful and we have the estimate  $\sup_{0 \leq s \leq t} E[|H_s|^p] < \infty$ , for any  $t > 0$ ,  $p \geq 2$ .

*Proof of Theorem 1.1.* We may assume that  $1 - \alpha/2 < \beta < 2 - \alpha$ . We will deduce the assertion from Theorem 5.3.

Since  $\alpha \geq 1$ , we can apply Theorem 4.3 and find a solution  $u_\lambda \in C_b^{1+\gamma}(\mathbb{R}^d, \mathbb{R}^d)$  to the resolvent equation (26) with  $\gamma = \alpha - 1 + \beta \in (0, 1)$ . By the last assertion of Theorem 4.3, we may choose  $\lambda$  sufficiently large in order that  $\|Du\|_0 = \|Du_\lambda\|_0 < 1/3$ . The crucial assumption about  $\gamma$  and  $\alpha$  in Theorem 5.3 is satisfied. Indeed  $2\gamma = 2\alpha - 2 + 2\beta > \alpha$  since  $\beta > 1 - \alpha/2$ . By Theorem 5.3 we obtain the result.  $\blacksquare$

*Remark 5.5.* Using the  $C^{\alpha+\beta}$ -regularity results proved in [28] one can show that the statement of Theorem 4.3 holds in the relevant case of  $\mathcal{L} = -(-\Delta)^{\alpha/2}$  even when  $0 < \alpha < 1$ . Therefore, by the same proof given in Section 5, one can prove that all the assertions of Theorem 1.1 hold even when  $0 < \alpha < 1$  and  $\beta > 1 - \frac{\alpha}{2}$  if the Lévy process  $L = (L_t)$  is a standard symmetric and rotationally invariant  $\alpha$ -stable process.

Nevertheless, it remains an open problem if the statements of Theorem 1.1 is true when  $0 < \alpha < 1$  for some other non-degenerate  $\alpha$ -stable processes  $L$  (note that the proof of [28] use the so-called extension property of  $-(-\Delta)^{\alpha/2}$  which is a typical property of such non-local operator). For instance, pathwise uniqueness is not clear when  $\beta \in (1 - \frac{\alpha}{2}, 1)$  and  $0 < \alpha < 1$  if  $L$  has the generator as in (3) and  $d > 1$ , i.e.,

$$\mathcal{L} = - \sum_{k=1}^d (-\partial_{x_k x_k}^2)^{\alpha/2}.$$

Note that for such operator we do not know if there exist regular solutions  $v$  to the non-local Kolmogorov equation (4) under the natural condition that  $b \in C_b^\beta(\mathbb{R}^d, \mathbb{R}^d)$ ,  $g \in C_b^\beta(\mathbb{R}^d)$  and  $\beta + \alpha > 1$ .

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